# INTERLACING FAMILIES AND RAMANUJAN GRAPHS

**[after MARCUS, SPIELMAN, AND SRIVASTAVA]**

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Abstract. Marcus, Spielman, and Srivastava used interlacing polynomials to prove the existence of bipartite Ramanujan graphs of all degrees, resolving a key problem in spectral graph theory. This write-up outlines their method, emphasizing the role of interlacing families in controlling eigenvalues and constructing optimal expander families.

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## 1. Introduction

1.1. **Universal covers and Ramanujan graphs.** Let $\Gamma := (V, E)$ be a finite graph. To define *Ramanujan graphs*, we make use of the concept of the *universal cover* of $\Gamma$. We begin with its definition:

**Definition 1.1.** *The universal cover of a graph $\Gamma$ is the unique infinite tree $\tilde{\Gamma}$ such that every connected lift of $\Gamma$ is a quotient of $\tilde{\Gamma}$.*

More concretely, fix a base vertex $v_0 \in V$. The vertices of the universal cover $\tilde{\Gamma}$ are given by *nonbacktracking walks* of the form $w = (v_0, v_1, \ldots, v_\ell)$ starting at $v_0$. Recall that such a walk satisfies $(v_i, v_{i+1}) \in E$ and $v_{i+1} \neq v_{i-1}$ for all $i$. Two vertices $w$ and $w'$ in $\tilde{\Gamma}$ are connected by an edge if one is a one-step extension (also known as *continuation*) of the other; that is, $w' = (v_0, \ldots, v_\ell, v_{\ell+1})$ or vice versa.

Let $A_{\tilde{\Gamma}}$ denote the *adjacency matrix* of the universal cover $\tilde{\Gamma} = (V_{\tilde{\Gamma}}, E_{\tilde{\Gamma}})$. This operator acts on the Hilbert space $\ell^2(V_{\tilde{\Gamma}})$ and is self-adjoint. The *spectral radius* $\rho(\tilde{\Gamma})$ of $A_{\tilde{\Gamma}}$ is defined as

$$\begin{aligned}\rho(\tilde{\Gamma}) &:= \sup_{\lambda \in \mathbb{C}} \{|\lambda| : A_{\tilde{\Gamma}} - \lambda \text{ is not invertible}\} \\ &= \sup_{\|x\|_2 = 1} \|A_{\tilde{\Gamma}} x\|_2,\end{aligned}$$

where $\|\cdot\|_2$ denotes the usual $\ell^2$ norm. The equality follows from the fact that $A_{\tilde{\Gamma}}$ is self-adjoint. We can also consider the adjacency matrix $A_\Gamma$ of the finite graph $\Gamma$. By the spectral theorem, all eigenvalues of $A_\Gamma$ are real, and we arrange them in non-increasing order:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\#V}.$$

If $\Gamma$ is bipartite, then its spectrum is symmetric about zero. The largest eigenvalue (and also the smallest, in the bipartite case) is considered *trivial*. We are now ready to define *Ramanujan graphs*.

**Definition 1.2.** *A graph $\Gamma$ is said to be* Ramanujan *if all of its nontrivial eigenvalues have absolute value at most the spectral radius of its universal cover.*

In [Gre95], it was shown that for every $\epsilon > 0$ and for any infinite family of graphs sharing the same universal cover $\tilde{\Gamma}$, all sufficiently large graphs in the family have a nontrivial eigenvalue at least $\rho(\tilde{\Gamma}) - \epsilon$. This highlights the difficulty of constructing infinite families of Ramanujan graphs with a common universal cover.

**Remark 1.3.** *Definition 1.2 generalizes the standard definition of Ramanujan graphs. For a $d$-regular graph, the usual condition is that all nontrivial eigenvalues have absolute value at most $2\sqrt{d-1}$, which is the spectral radius of its universal cover.*

### 1.2. Explicit constructions and related works.

Infinite families of constant-degree Ramanujan graphs were first constructed by Lubotzky, Phillips, and Sarnak in [LPS88], and independently by Margulis in [Mar88]. The construction in [LPS88] uses the Ramanujan–Petersson bounds (proved by Deligne) for Hecke eigenvalues to build Ramanujan graphs from Cayley graphs. These are $(p+1)$-regular graphs for primes $p$. Later, Morgenstern extended this to all prime powers $q$ by constructing infinite families of $(q+1)$-regular Ramanujan graphs in [Mor94]. Further related constructions can be found in [JL97], [Chi92], [Piz90], and [MSS18].

### 1.3. Main result.

Suppose $\Gamma$ is a finite $d$-regular graph. Then its universal cover is the infinite $d$-regular tree (see [LS96]). We now state the first main theorem.

**Theorem 1.4** (Theorem 5.5, [MSS15])**.** *For every $d \geq 3$, there exists an infinite sequence of $d$-regular bipartite Ramanujan graphs.*

Recall that a $(c, d)$-biregular bipartite graph is a bipartite graph in which every vertex on one side of the bipartition has degree $c$, and every vertex on the other side has degree $d$. The universal cover of such a graph is the infinite $(c, d)$-biregular tree, where vertex degrees alternate between $c$ and $d$ across successive levels (see [LS96]). We now state the second main theorem.

**Theorem 1.5** (Theorem 5.6, [MSS15])**.** *For all $c, d \geq 3$, there exists an infinite sequence of $(c, d)$-biregular bipartite Ramanujan graphs.*

### 1.4. Overview and outline of the proof.

The authors use an inductive strategy, which we now briefly describe. Starting from a Ramanujan graph $\Gamma$, they construct an infinite sequence of graphs by repeatedly taking 2-*lifts* (see Section 2). The core idea is to show that among the 2-lifts of $\Gamma$, at least one remains Ramanujan.

This requires controlling the eigenvalues of certain associated random matrices. The authors accomplish this using the method of *interlacing polynomials*. Section 2 introduces preliminary results, while Section 3 covers the method of interlacing polynomials. The complete proofs of Theorems 1.4 and 1.5 are given in Section 4.

## 2. Preliminaries

### 2.1. 2-lifts.

Let $\Gamma := (V, E)$ be a graph. A graph $\hat{\Gamma} := (\hat{V}, \hat{E})$ is called a *2-lift* of $\Gamma$ if there exists a $2:1$ covering map $\pi : \hat{\Gamma} \to \Gamma$. We can describe 2-lifts more concretely as follows: for each vertex $v \in V$, there are two corresponding vertices $\{v_0, v_1\} \subset \hat{V}$, known as the *fiber* of $v$. For every edge $(v, w) \in E$, there are two corresponding edges in $\hat{E}$. If $\{v_0, v_1\}$ and $\{w_0, w_1\}$ are the fibers of $v$ and $w$, respectively, then $\hat{E}$ must contain either the pair $\{(v_0, w_0), (v_1, w_1)\}$ or the pair $\{(v_0, w_1), (v_1, w_0)\}$ (see Figure 2.1 below).

Therefore, any 2-lift of $\Gamma$ can be described using a *signing* map $s : E \to \{\pm 1\}$, which determines the type of edges in the 2-lift. Specifically, we assign $s(e) = 1$ for Type I edges and $s(e) = -1$ for Type II edges (using the notation of Figure 2.1). It is clear that the 2-lifts of $\Gamma$ are in one-to-one correspondence with signing maps of $\Gamma$.

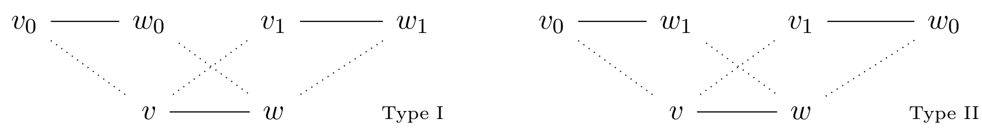



FIGURE 1. Edges in 2-lifts.

**Definition 2.1.** *Let $\Gamma := (V, E)$ be a graph, and let $s : E \to \{\pm 1\}$ describe a 2-lift. The* signed adjacency matrix $A_s$ *of the 2-lift is a square matrix of size $\#V$ with entries:*

$$A_s(v, w) := s(v, w) \quad \textit{for all } v, w \in V.$$

Bilu and Linial studied 2-lifts in [BL06], using them to construct large expanders of fixed degree. We will need the following key lemma from their work.

**Lemma 2.2** (Lemma 3.1, [BL06])**.** *The eigenvalues of a 2-lift of $\Gamma$ consist of the eigenvalues of $\Gamma$, together with the eigenvalues of the corresponding signed adjacency matrix (with multiplicity).*

The key idea is to control the eigenvalues of a signed adjacency matrix and iteratively apply this process to construct arbitrarily large Ramanujan graphs of fixed degree.

2.2. **Matching polynomials.** Recall that a *matching* in a graph $\Gamma := (V, E)$ is a subset of $E$ such that no two edges share a common vertex. For $i \in \mathbb{N}$, denote by $m_i$ the number of matchings in $\Gamma$ of size $i$. The *matching polynomial* $\mu_\Gamma$ of $\Gamma$ is defined as

$$\mu_\Gamma(t) := \sum_{i \in \mathbb{N}} (-1)^i m_i \cdot t^{\#V - 2i}.$$

Note that this is indeed a polynomial of degree $\#V$, since $m_i = 0$ for $i > \#V/2$. Recall that simple walks are walks on graphs with all vertices distinct. We define *path trees.*

**Definition 2.3.** *Given a graph $\Gamma$ and a vertex $v$, the path tree $P(\Gamma, v)$ contains one vertex for every simple walk in $\Gamma$ beginning at $v$. Two vertices in $P(\Gamma, v)$ share an edge if the simple walk corresponding to one is a continuation of the simple walk corresponding to the other.*

We now state the following fundamental property of matching polynomials, due to Godsil, which plays a crucial role in the proof of the main result.

**Theorem 2.4** ([God81])**.** *Let $P(\Gamma, v)$ be a path tree of $\Gamma$. Then the matching polynomial of $\Gamma$ divides the characteristic polynomial of the adjacency matrix of $P(\Gamma, v)$. In particular, all roots of $\mu_\Gamma(t)$ are real and have absolute value at most $\rho(P(\Gamma, v))$.*

Let $\tilde{\Gamma}$ be the universal cover of $\Gamma$. Using the notation from the above theorem, we observe that $P(\Gamma, v)$ is a finite induced subgraph of $\tilde{\Gamma}$. Therefore, $\rho(P(\Gamma, v)) \leq \rho(\tilde{\Gamma})$, from which we deduce the following corollary.

**Corollary 2.5.** *The roots of $\mu_\Gamma(t)$ are bounded in absolute value by $\rho(\tilde{\Gamma})$.*

The following is a key result that relates the characteristic polynomials of signed adjacency matrices of $\Gamma$ to the matching polynomial $\mu_\Gamma$. Consider the set

$$S_\Gamma := \{A_s \mid s : E \to \{\pm 1\}, \text{ where } A_s \text{ denotes the associated signed adjacency matrix}\},$$

equipped with the uniform probability measure. Define $X$ as the random variable on $S_\Gamma$ that outputs the characteristic polynomial of an element in $S_\Gamma$. Then we have the following result.

**Theorem 2.6** ([GG78])**.** *In the above setting, we have*

$$\mathbb{E}[X] = \mu_\Gamma.$$

*Proof.* Let $\mathrm{sym}(S)$ denote the set of permutations of a set $S$ and let $|\pi|$ denote the number of inversions of a permutation $\pi$. For convenience, let $n = \#V_\Gamma$. We expand the determinant as a sum over permutations $\sigma \in \mathrm{sym}([n])$ and use linearity of expectation, we have (recalling that $A_s$ are chosen uniformly at random as signings are chosen uniformly at random)

$$\mathbb{E}[\det(xI - A_s)] = \mathbb{E}\left[\sum_{\sigma \in \mathrm{sym}([n])} (-1)^{|\sigma|} \prod_{i=1}^{n} (xI - A_s)_{i,\sigma(i)}\right].$$

To obtain the coefficients in the expected polynomial, we group terms based on the number of non-fixed points in the permutation $\sigma$. Let $S \subset [n]$ be the set of indices such that $\sigma(i) \neq i$, and let $\pi$ be the restriction of $\sigma$ to $S$. Then we obtain that the expectation is:

$$\sum_{k=0}^{n} x^{n-k} \sum_{S \subset [n], |S|=k} \sum_{\pi \in \mathrm{sym}(S)} \mathbb{E}\left[(-1)^{|\pi|} \prod_{i \in S} (A_s)_{i,\pi(i)}\right],$$

where $\pi$ denotes the part of $\sigma$ with $\sigma(i) \neq i$. Next, we note in the above expression that the expectation of $s_{i,j}$ vanishes unless each of them appears with even multiplicity. Therefore, the only permutations $\pi$ that contribute are involutive matchings, i.e., perfect matchings consisting solely of disjoint transpositions. These only exist when $|S|$ is even; in that case, each matching contributes $(-1)^{|S|/2}$, since each transposition has one inversion, and there are $|S|/2$ such transpositions. Since $\mathbb{E}[s_{ij}^2] = 1$, we are left with

$$\begin{aligned}\mathbb{E}[\det(xI - A_s)] &= \sum_{k=0}^{n} x^{n-k} \sum_{|S|=k} \sum_{\text{matching } \pi \text{ on } S} (-1)^{|S|/2} \cdot 1 \\ &= \mu_\Gamma(x).\end{aligned}$$

This concludes the proof. □

2.3. **Real stable polynomials.** We begin this section by defining *real stable polynomials.*

**Definition 2.7.** *A polynomial $f \in \mathbb{R}[t_1, \dots, t_n]$ is called* real stable *if it is either identically zero or satisfies*

$$f(z_1, \dots, z_n) \neq 0$$

*for all $(z_1, \dots, z_n) \in \mathbb{H}^n$.*

Let $\mathcal{S}[t_1, \dots, t_n] \subset \mathbb{R}[t_1, \dots, t_n]$ denote the set of real stable polynomials and set

$$\hat{\mathcal{S}} := \cup_{n \geq 1} \mathcal{S}[t_1, \dots, t_n].$$

Note that the set $\hat{\mathcal{S}}$ is closed under multiplication, and a univariate real stable polynomial is real rooted. Next, we present some useful real stable polynomials and describe operators that preserve the set of real stable polynomials. First, we introduce a notation. For any $c \in \mathbb{R}$, we define the operator $Z_{t_i}^c : \mathbb{R}[t_1, \dots, t_n] \to \mathbb{R}[t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n]$, which acts by setting $t_i = c$.

**Lemma 2.8.** *In the above setting, the following properties hold:*

1. *Let $A_1, \dots, A_n$ be positive semidefinite matrices. We have that*

$$\det(t_1 A_1 + \cdots + t_n A_n) \in \hat{\mathcal{S}}.$$

2. *We have that $Z_{t_i}^c(\mathcal{S}[t_1, \dots, t_n]) \subset \mathcal{S}[t_1, \dots, t_n])$. In other words the operators $Z_{t_i}^c$ preserve the set of real stable polynomials.*

*Proof.* The statement of *(1)* follows from [BB08][Proposition 6.4].

For *(2)*, we assume $i = n$. Suppose $f$ is a nonzero real stable polynomial. Consider the sequence of polynomials $(f(t_1, \ldots, c + 2^{-k}\iota))_k$. It is clear that every polynomial in this sequence is nonvanishing on $\mathbb{H}^{n-1}$. Hence, by Hurwitz's theorem [RS02][Theorem 1.3.8], the limit is either zero or nonvanishing on $\mathbb{H}^{n-1}$. □

As usual, let $\partial_{t_i}$ denote the operator which acts by partial differentiation with respect to $t_i$. For $\alpha, \beta \in \mathbb{N}^n$ we define:

$$t^\alpha := \prod_{i \leq n} t_i^{\alpha_i} \text{ and } \partial^\beta := \prod_{i \leq n} \partial_{t_i}^{\beta_i}.$$

We have a result on operators preserving the set $\hat{\mathcal{S}}$.

**Theorem 2.9** (Theorem 1.3, [BB10])**.** *Let $T : \mathbb{R}[t_1, \ldots, t_n] \to \mathbb{R}[t_1, \ldots, t_n]$ be an operator of the form*

$$T = \sum_{\alpha, \beta \in \mathbb{N}^n} c_{\alpha,\beta} t^\alpha \partial^\beta$$

*where $c_{\alpha,\beta} \in \mathbb{R}$ and $c_{\alpha,\beta}$ is zero for all but finitely many terms. Define*

$$F_T(t, u) := \sum_{\alpha,\beta} c_{\alpha,\beta} t^\alpha u^\beta.$$

*Then $T(\mathcal{S}[t_1, \ldots, t_n]) \subset \mathcal{S}[t_1, \ldots, t_n]$ if and only if $F_T(t, -u) \in \hat{\mathcal{S}}$.*

We now present the following key lemma.

**Lemma 2.10.** *In the previous setting, the following hold:*

*(1) For all $a, b \in \mathbb{R}_{\geq 0}$, the operator $1 - a\partial_{t_i} - b\partial_{t_j}$ preserves the set of real stable polynomials for all $i, j$.*

*(2) For an invertible matrix $A$, vectors $u, v$, and $p \in [0, 1]$,*

$$\begin{aligned} Z_x^0 Z_y^0 (1 + p\partial_x + (1-p)\partial_y) \det(A + xuu^T + yvv^T) \\ = p \det(A + uu^T) + (1-p) \det(A + vv^T). \end{aligned}$$

*Proof.* For *(1)* we use Theorem 2.9 to deduce the operator $1 - a\partial_{t_i} - b\partial_{t_j}$ preserves real stable polynomials if and only if $1 - a \cdot u - b \cdot v$ is a real stable polynomial for all $a, b \in \mathbb{R}_{\geq 0}$. This is clearly true.

For *(2)*, we use the matrix determinant lemma. In our setting, it states that for every real number $x$,

$$\det(A + xuu^T) = \det(A)(1 + xu^T A^{-1} u).$$

Differentiating with respect to $x$:

$$\partial_x \det(A + xuu^T) = \det(A)(u^T A^{-1} u).$$

Using the above formula, we deduce that

$$Z_x^0 Z_y^0 (1+p\partial_x+(1-p)\partial_y) \det(A + xuu^T + yvv^T) = \det(A) \left(1 + p(u^T A^{-1} u) + (1-p)(v^T A^{-1} v)\right).$$

Now again using the matrix determinant lemma, the right-hand side is,

$$p \det(A + uu^T) + (1-p) \det(A + vv^T).$$

□

As a consequence of Lemma 2.8 and Lemma 2.10, we obtain the following theorem.

**Theorem 2.11** (Theorem 6.6, [MSS15]). *Let $u_1, \ldots, u_m, v_1, \ldots, v_m$ be vectors in $\mathbb{R}^n$, let $p_1, \ldots, p_m$ be real numbers in $[0,1]$, and let $D$ be a positive semidefinite matrix. Then the polynomial*

$$P(t) := \sum_{S \subset [m]} \prod_{j \in S} p_j \prod_{j \notin S} (1-p_j) \det\left(t + D + \sum_{j \in S} u_j u_j^T + \sum_{j \notin S} v_j v_j^T\right)$$

*is real rooted.*

*Proof.* We briefly outline the proof. Since $P$ is univariate, it suffices to show that $P$ is real stable. Consider the polynomial

$$Q(t, x_1, \ldots, x_m, y_1, \ldots, y_m) := \det\left(t + D + \sum_j x_j u_j u_j^T + \sum_j y_j v_j v_j^T\right).$$

By Lemma 2.8, we deduce that $Q$ is real stable. Next, using induction, we will prove that

$$P(t) = \left[\prod_{j=1}^{m} Z_{x_j}^0 Z_{y_j}^0 T_j\right] Q,$$

where $T_j := 1 + p_j \partial_{x_i} + (1-p_j)\partial_{y_j}$. We claim that for each $k \leq m$,

$$\left(\prod_{j=1}^{k} Z_{x_j}^0 Z_{y_j}^0 T_j\right) Q(t, x_1, \ldots, x_m, y_1, \ldots, y_m)$$

equals

$$\sum_{S \subseteq [k]} \left(\prod_{j \in S} p_j\right) \left(\prod_{j \in [k] \setminus S} 1 - p_j\right) \det\left(tI + D + \sum_{j \in S} u_j u_j^T + \sum_{j \in [k] \setminus S} v_j b_j^T + \sum_{j > k} x_j u_j u_j^T + y_j v_j v_j^T\right).$$

The base case $k = 0$ is clear. We deduce the inductive step using Lemma 2.10 part *(2)*. Next, using Lemma 2.8 and Lemma 2.10, we obtain that $P$ is real stable and consequently real rooted. □

## 3. Interlacing families

Our goal is to analyze the eigenvalues of signed adjacency matrices. By Theorem 2.4 and Corollary 2.5, we can derive bounds on the roots of the matching polynomial. Furthermore, the matching polynomial is expressed as a weighted sum of the characteristic polynomials of signed adjacency matrices. In general, the roots of a sum of polynomials do not necessarily provide information about the roots of the individual summands. However, in our setting, we can establish the desired results by leveraging the notion of *interlacing families*.

**Definition 3.1.** *We say that a polynomial $g(x) = \prod_{i=1}^{n-1}(x - \alpha_i)$* interlaces *a polynomial $f(x) = \prod_{i=1}^{n}(x - \beta_i)$ if*

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \cdots \leq \alpha_{n-1} \leq \beta_n$$

*We say that polynomials $f_1, \ldots, f_k$ have a* common interlacing *if there is a polynomial $g$ so that $g$ interlaces $f_i$ for each $1 \leq i \leq k$.*

We present a simple criterion to determine whether a given collection of polynomials admits a common interlacing. See [CS07], [Ded92], or [Fel80] for a proof.

**Lemma 3.2.** *Let $f_1, \ldots, f_k$ be polynomials of the same degree with positive leading coefficients. Then $f_1, \ldots, f_k$ have a common interlacing if and only if $\sum_{i=1}^{k} \lambda_i f_i$ is real rooted for all convex combinations $\lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1$.*

We are now set to define interlacing families.

**Definition 3.3.** *For every assignment $s_1, \ldots, s_m \in \{\pm 1\}$ let $f_{s_1,\ldots,s_m}(x)$ be a real rooted degree $n$ polynomial with positive leading coefficient. For a partial assignment $s_1, \ldots, s_k \in \{\pm 1\}$ with $k < m$, define*

$$f_{s_1,\ldots,s_k} := \sum_{s_{k+1},\ldots,s_m \in \{\pm 1\}} f_{s_1,\ldots,s_k,s_{k+1},\ldots,s_m},$$

*as well as*

$$f_{\emptyset} := \sum_{s_1,\ldots,s_m \in \{\pm 1\}} f_{s_1,\ldots,s_m}.$$

*We say that the polynomials $\{f_{s_1,\ldots,s_m}\}_{s_1,\ldots,s_m}$ form an interlacing family if for all $k = 0, \ldots, m-1$, and all $s_1, \ldots, s_k \in \{\pm 1\}^k$, the polynomials*

$$\{f_{s_1,\ldots,s_k,t}\}_{t \in \{\pm 1\}}$$

*have a common interlacing.*

Another way to understand interlacing families is using *rooted binary trees* with the root $f_\emptyset$ along with the property that the polynomial on each node is the sum of the two polynomials on the child nodes, and siblings have common interlacing. For example, in the case $m = 2$:

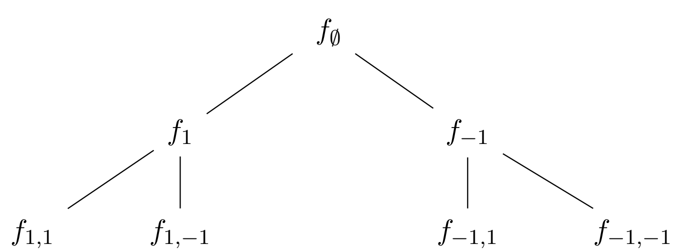
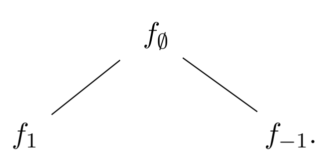



FIGURE 2. Associated binary tree in the case $m = 2$.

We now state a key theorem on interlacing families, which will play a crucial role in the proof of the main result.

**Theorem 3.4** (Theorem 4.4, [MSS15]). *Let $\{f_{s_1,\ldots,s_m}\}$ be an interlacing family of polynomials. Then, there exists some $s_1, \ldots, s_m \in \{\pm 1\}$ so that the largest root of $f_{s_1,\ldots,s_m}$ is at most the largest root of $f_\emptyset$.*

*Proof.* We induct on the index $m$. In case $m = 1$, we have that $f_\emptyset = f_1 + f_{-1}$ and the tree is




Let $g$ be a polynomial that interlaces both $f_1$ and $f_{-1}$, and let $\beta_{n-1}$ denote its largest root. Then each of $f_1$ and $f_{-1}$ has at most one root greater than or equal to $\beta_{n-1}$. Moreover, since both $f_1$ and $f_{-1}$ have positive leading coefficients, they are positive for large values of $x$, and so $f_\emptyset(x) > 0$ for sufficiently large $x$. In particular, we have $f_1(\beta_{n-1}) \leq 0$ and $f_{-1}(\beta_{n-1}) \leq 0$, and hence

$$f_\emptyset(\beta_{n-1}) = f_1(\beta_{n-1}) + f_{-1}(\beta_{n-1}) \leq 0.$$

Therefore, by the intermediate value theorem, there exists some $\alpha \geq \beta_{n-1}$ such that $f_\emptyset(\alpha) = 0$. Since $f_\emptyset(\alpha) = f_1(\alpha) + f_{-1}(\alpha)$, it follows that for some $i \in \{\pm 1\}$, we must have $f_i(\alpha) \geq 0$. This concludes the proof in the base case $m = 1$.

Now suppose the theorem holds for all $k < m$, and consider the case $k = m$. Again, we have

$$f_\emptyset = f_1 + f_{-1}.$$

Then, for some $i \in \{\pm 1\}$, the largest root of $f_i$ is at most the largest root of $f_\emptyset$. By the inductive hypothesis, applied to the subtree rooted at $f_i$ (which has depth $m-1$), there exist signs $s_1, s_2, \ldots, s_m \in \{\pm 1\}$ such that the largest root of $f_{s_1,s_2,\ldots,s_m}$ is at most the largest root of $f_\emptyset$. This completes the inductive step.

□

## 4. Proofs of the Main Theorems

Let $\Gamma = (V, E)$ be a finite graph. Note that after choosing an ordering of the edges, we identify the signing of $\Gamma$ with tuples $s \in \{\pm 1\}^m$ where $m = \#E$. For any $s \in \{\pm 1\}^m$ let $f_s$ denote the characteristic polynomial of the corresponding signed adjacency matrix. We begin by proving the following result, which suffices to prove Theorem 1.4 and 1.5.

**Theorem 4.1** (Theorem 5.1, [MSS15]). *Let $p_1, \ldots, p_m$ be numbers in $[0,1]$. Then, the following polynomial is real rooted*

$$\sum_{s\in\{\pm 1\}^m} \left(\prod_{i:s_i=1} p_i\right) \left(\prod_{i:s_i=-1} (1-p_i)\right) f_s(x).$$

*Proof.* Let $v \in V$ be any vertex, and let $e_v$ denote the standard basis vector in the direction of $v$. Let $d_v$ be the degree of the vertex $v$, and define $d := \max_{v\in V} d_v$ to be the maximum degree in $\Gamma$. Let $D$ be the diagonal matrix of size $\#V$, where the $v$-th diagonal entry is $d - d_v$ (we fix an ordering on the vertex set $V$). For each pair $u, v \in V$, define the vectors $a_{u,v} := e_u - e_v$ and $b_{u,v} := e_u + e_v$.

Using Theorem 2.7, we deduce that the following polynomial is real-rooted:

$$\sum_{s\in\{\pm 1\}^m} \left(\prod_{i:s_i=1} p_i\right) \left(\prod_{i:s_i=-1} (1-p_i)\right) \det\left(xI + D + \sum_{s_{u,v}=1} a_{u,v}a_{u,v}^T + \sum_{s_{u,v}=-1} b_{u,v}b_{u,v}^T\right).$$

Further, a direct computation shows that

$$D + \sum_{s_{u,v}=1} a_{u,v}a_{u,v}^T + \sum_{s_{u,v}=-1} b_{u,v}b_{u,v}^T = dI - A_s.$$

Now, substituting this into the expression above, we see that the polynomial

$$\sum_{s\in\{\pm 1\}^m} \left(\prod_{i:s_i=1} p_i\right) \left(\prod_{i:s_i=-1} (1-p_i)\right) \det\left(xI + dI - A_s\right)$$

is real-rooted. This concludes the proof.

□

We have the following important corollary.

**Corollary 4.2** (Theorem 5.2, [MSS15]). *The polynomials $\{f_s\}_{s\in\{\pm 1\}^m}$ form an interlacing family.*

*Proof.* Let $0 \leq k \leq m-1$ and fix any $s_1, \ldots, s_k \in \{\pm 1\}$, and let $\lambda \in [0,1]$. In Theorem 4.1 plugging in $p_{k+1} = \lambda$, $p_{k+2}, \ldots, p_m = 1/2$, and $p_i = (1+s_i)/2$ for $1 \leq i \leq k$ we obtain that the polynomial

$$\lambda f_{s_1,\ldots,s_k,1}(x) + (1-\lambda) f_{s_1,\ldots,s_k,-1}(x)$$

is real rooted. Using Lemma 3.2 we deduce $f_{s_1,\dots,s_k,1}, f_{s_1,\dots,s_k,-1}$ have a common interlacing. Hence the polynomials $\{f_s\}_{s\in\{\pm1\}^m}$ form an interlacing family. □

We are now ready to prove the main results.

4.0.1. *Proof of Theorem 1.4.* Let $d \geq 3$ and let $T_d$ denote the infinite $d$-regular tree. Let $\Gamma_{d,d}$ be the complete bipartite $d$-regular graph. Note that all 2-lifts of a bipartite graph remain bipartite, the spectrum of a bipartite graph is symmetric about zero, and $\Gamma_{d,d}$ is a Ramanujan graph since all its nontrivial eigenvalues are zero. Moreover, its universal cover satisfies $\tilde{\Gamma}_{d,d} \simeq T_d$.

By Theorem 2.6, the expected characteristic polynomial over all signings of $\Gamma_{d,d}$ is equal to its matching polynomial $\mu_{\Gamma_{d,d}}$. Corollary 4.2 then tells us that the collection $\{f_s\}_s$ of characteristic polynomials corresponding to different signings forms an interlacing family. Applying Theorem 3.4, we deduce that there exists a signing $s$ such that the largest root of $f_s$ is at most the largest root of $\mu_{\Gamma_{d,d}}$.

Finally, Corollary 2.5 implies that the largest root of $\mu_{\Gamma_{d,d}}$ is at most the spectral radius $\rho(T_d)$. Therefore, by Lemma 2.2, there exists at least one 2-lift of $\Gamma_{d,d}$ which is a $d$-regular bipartite Ramanujan graph with twice as many vertices. Iterating this process inductively yields an infinite family of $d$-regular bipartite Ramanujan graphs, completing the proof of the theorem.

□

Although the proof of Theorem 1.5 is largely analogous to the one given above, we present it here in full for the sake of completeness.

4.0.2. *Proof of Theorem 1.5.* Let $c, d \geq 3$, and let $T_{c,d}$ denote the infinite $(c,d)$-biregular tree, in which vertex degrees alternate between $c$ and $d$ at successive levels. Let $\Gamma_{c,d}$ be the complete bipartite $(c,d)$-biregular graph. Note that all of its 2-lifts remain bipartite and $(c,d)$-biregular. Moreover, $\Gamma_{c,d}$ is a Ramanujan graph: it is straightforward to verify that its trivial eigenvalues are $\pm\sqrt{cd}$, and since the adjacency matrix has rank 2, all nontrivial eigenvalues are zero. Its universal cover satisfies $\tilde{\Gamma}_{c,d} \simeq T_{c,d}$.

By Theorem 2.6, the expected characteristic polynomial over all signings of $\Gamma_{c,d}$ is equal to its matching polynomial $\mu_{\Gamma_{c,d}}$. Then, by Corollary 4.2, the collection $\{f_s\}_s$ of characteristic polynomials corresponding to all signings forms an interlacing family. Applying Theorem 3.4, we conclude that there exists a signing $s$ such that the largest root of $f_s$ is at most the largest root of $\mu_{\Gamma_{c,d}}$. Finally, Corollary 2.5 ensures that the largest root of $\mu_{\Gamma_{c,d}}$ is at most the spectral radius $\rho(T_{c,d})$.

Therefore, by Lemma 2.2, there exists at least one 2-lift of $\Gamma_{c,d}$ which is a $(c,d)$-biregular bipartite Ramanujan graph with twice as many vertices. Repeating this argument inductively yields an infinite family of $(c,d)$-biregular bipartite Ramanujan graphs, completing the proof.

□

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