TITCHMARSH DIVISOR PROBLEM IN INTERVALS

HRISHABH MISHRA

Abstract. We prove an approximate formula for the sum $\sum_{n\le x}\Lambda(n)\tau(n-1)$ involving zeros of Dirichlet L-functions similar to the classical approximate formula for $\sum_{n\le x}\Lambda(n)$. Using the approximate formula we obtain asymptotics for $\sum_{x<n\le x+y}\Lambda(n)\tau(n-1)$ where $x^\theta<y<x$ for suitable $0<\theta<1$ without assuming the Riemann hypothesis for Dirichlet L-functions.

1. Introduction

Many important problems in analytic number theory can be regarded as problems in understanding correlations of pairs of arithmetic functions. Quantitatively, solving a number of important problems involves the knowledge of the asymptotic behavior of the sum

$$\sum_{n\le x} f(n)g(n-1), \tag{1.1}$$

for certain arithmetic functions $f,g:\mathbb{N}\to\mathbb{C}$. For instance, the twin-prime conjecture can be regarded as a conjecture of the sum 1.1 with suitable f,g. In general, obtaining asymptotic behavior for the sum 1.1 is a very hard problem but we know the asymptotic behavior in certain cases. We understand the asymptotics well when g is the function τ counting the number divisors and f is either the indicator function of primes [Fou85, BFI86], the function τ_k counting the number of ways a number can be written as a product of k positive integers [Mot80, Top16, Top17] or the indicator function of integers with small prime factors [FT90, Dra15]. When g is the function τ, the full asymptotic expansion for the sum 1.1 was obtained by Drappeau and Topacogullari in [DT19] for a large class of multiplicative functions f that are periodic over primes. One can also consider the weighted correlations, in particular, we can consider the *logarithmically weighted* correlation

$$\sum_{n\le x}\frac{f(n)g(n-1)}{n}. \tag{1.2}$$

This has been studied in great detail in Tao's paper [Tao16]. One important case of the problem 1.1 is the Titchmarsh divisor problem, the special case when f is the Von Mangoldt function $\Lambda:\mathbb{N}\to\mathbb{C}$ defined below

$$\Lambda(n):=\begin{cases}\log p & \text{if } n=p^\alpha \text{ for some prime } p,\\ 0 & \text{otherwise.}\end{cases}$$

and g is the function τ. This problem was first considered by Titchmarsh in 1930 [Tit30]. Let us denote the sum 1.1 in this special case by $T(x)$. Assuming the Riemann hypothesis for all Dirichlet L-functions Titchmarsh proved that $T(x)\sim C_1x\log x$ for some $C_1>1$. The same asymptotic for $T(x)$ was proved by Linnik [Lin63] unconditionally using the dispersion method. Simpler proofs were given by Rodriquez [Rod65] and Halberstam [Hal67] using Bombieri-Vinogradov and Brun-Titchmarsh theorems. A much better estimate was obtained by Fouvry [Fou85] and Bombieri-Freidlander-Iwaniec [BFI86] independently, we state the formula below.

Theorem A (Fouvry [Fou85], Bombieri-Friedlander-Iwaniec [BFI86])**.** *For all $A > 0$ and $x \geq 3$,*

$$T(x) = C_1 x\{\log x + 2\gamma - 1 - 2C_2\} + O_A\left(\frac{x}{\log^A x}\right)$$

where $C_1 = \prod_p \left(1 + \frac{1}{p(p-1)}\right)$, $C_2 = \sum_p \left(\frac{\log p}{1+p(p-1)}\right)$ and γ denotes the Euler-Mascheroni constant.

The proof of this result involves obtaining Bombieri-Vinogradov-type estimates without absolute values in the sum. Bombieri-Friedlander-Iwaniec prove the following estimate for primes in arithmetic progressions.

Theorem B ([BFI86])**.** *For every $A > 0$ we can find B and C depending on A such that the inequality*

$$\left| \sum_{q \leq Q, (q,a)=1} \pi(x; q, a) - \frac{1}{\phi(q)}\pi(x) \right| \leq \frac{Cx}{\log^A 3x}$$

holds for all $x \geq 1$, $Q \leq x/\log^B 3x$ and any integer $1 \leq |a| \leq \log^A 3x$.

Further in 2015, Drappeau [Dra17, Theorem 1.1] obtained a power-saving error term in the above estimate assuming the Riemann hypothesis for all Dirichlet L-functions. We state the theorem below.

Theorem C (Drappeau [Dra17])**.** *Assume GRH. Then for some $\delta > 0$ and all $x \geq 2$,*

$$T(x) = C_1 x\{\log x + 2\gamma - 1 - 2C_2\} + O(x^{1-\delta}).$$

We elaborate more on his method in Section 2 below. He also obtains an unconditional estimate [Dra17, Theorem 1.2]. As a corollary [Dra17, Corollary 1.4] of this estimate, he proves the following unconditional inequality

Corollary (Drappeau [Dra17])**.** *With an effective implicit constant, we have*

$$\sum_{p \leq x} \tau(p-1) \leq C_1\{x + 2\mathrm{li}(x)(\gamma - C_2)\} + O(xe^{-\delta\sqrt{x}})$$

for some $\delta > 0$.

Instead of considering sums of the form 1.1, in this paper we consider the asymptotic behavior of sums of the form

$$\sum_{x < n \leq x+y} f(n)g(n-1), \tag{1.3}$$

for arithmetic functions f, g and $\alpha(x) \leq y < x$ where $\alpha(x) : \mathbb{R} \to \mathbb{R}$ is a suitable function. We consider the special case when $f := \Lambda$ and $g := \tau$. A similar problem was considered by Assing, Blomer, and Li in [ABL21, cf. Theorem 1.1]. We state the main theorems of this paper.

1.1. **Main Results.** We obtain an approximate formula for the sum $T(x)$ involving zeros of Dirichlet L-functions similar to other approximate formulas in multiplicative number theory. We state the formula below as Theorem 1.1. To state the result, let us denote

$$L_{R,q}(s) := \prod_{\substack{\chi \bmod q, \\ cond(\chi)\le R}} L(s,\chi).$$

Theorem 1.1. *There exist $A>0$ and $\delta\in(0,1)$ such that for $T\ge 2$ and all $x\ge 2$ we have that*

$$T(x) = C_1x\{\log x + 2\gamma - 1 - 2C_2\} - \sum_{q\le\sqrt{x},}\ \sum_{\substack{L_{R,q}(\rho)=0 \\ |\mathrm{Im}(\rho)|\le T}} \frac{2(x^\rho - q^{2\rho})}{\phi(q)\rho}$$
$$+ O\left(R\frac{x}{T}\log^3(xT) + R\log^3 x + x(\log x)^A R^{-1/9} + x^{1-\delta}\right)$$

where $1\le R\le x^{1/10}$ is a parameter.

Next, using the above approximate formula, zero-free regions, and zero-density results about Dirichlet L-functions we obtain an estimate for the sum 1.3 in the special we are considering with $\alpha(x) := x^\theta$ for suitable $\theta\in(0,1)$. More precisely, we prove the following estimate.

Theorem 1.2. *For some $1>\theta>0$ and for all $x\gg 2$ and $x^\theta<y<x$ the following estimate holds,*

$$T(x+y) - T(x) = C_1y\log x + C_1(x+y)\log\left(1+\frac{y}{x}\right) + \{2\gamma-1-2C_2\}y + O\left(\frac{y}{\log x}\right).$$

Next, we describe our notation and give an outline of the paper.

1.2. **Notation and Outline.** We use the notation widely used in literature. We define the following functions from $\mathbb{R}$ to $\mathbb{C}$,

$$\psi(x) := \sum_{1<n\le x}\Lambda(n),\ \ \psi(x;q,a) := \sum_{\substack{n\le x \\ n\equiv a \bmod q}}\Lambda(n),\ \ \psi_q(x) := \sum_{\substack{n\le x \\ (n,q)=1}}\Lambda(n).$$

Let $\chi:\mathbb{Z}\to\mathbb{C}$ be a Dirichlet character then we define the function $\psi(x,\chi):\mathbb{R}\to\mathbb{C}$,

$$\psi(x,\chi) := \sum_{1<n\le x}\Lambda(n)\chi(n),$$

and we define the functions $\phi_0(x), \phi_0(x,\chi):\mathbb{R}\to\mathbb{C}$ below

$$\psi_0(x) := \frac{\psi(x^+)+\psi(x^-)}{2},\ \ \psi_0(x,\chi) := \frac{\psi(x^+,\chi)+\psi(x^-,\chi)}{2}.$$

This paper consists of four sections including the Introduction. In Section 2, we use standard results to prove some approximate formulas for suitable sums. Then we recall some setup from previous work on the Titchmarsh divisor problem in [Dra17] and reduce the proof of Theorem 1.1 to obtain an approximate formula for the sum S_1^- introduced in [Dra17, Section 6] and defined in Section 2. We obtain the approximate formula for S_1^- in Section 3 and prove Theorem 1.1. Entire Section 4 is devoted to the proof of Theorem 1.2 using Theorem 1.1 and results about zeros of the Dirichlet L-functions stated in Section 2.

2. Preliminaries

First, we state an approximate formula for $\psi(x,\chi)$ which can be easily obtained from the results in [MV06, Chapter 12]. We will use this formula to obtain an approximate formula for the sum in the Titchmarsh divisor problem. For principal characters, we have the following result,

Lemma 2.1. *Let $T \geq 2$ and let χ_0 be a principal Dirichlet character modulo $q \geq 1$. Then we have the following,*
(2.1)
$$\psi(x,\chi_0) = x - \sum_{\substack{L(\rho,\chi_0)=0\\ |\mathrm{Im}(\rho)|\leq T}} \frac{x^\rho}{\rho} - \log 2\pi + O\left(\log x \log q + \log x + \frac{x}{T}(\log xT)^2 + \log q \log\left(\frac{T\log q}{2\pi}\right)\right).$$

Proof. For $q=1$ this follows immediately from [MV06, Theorem 12.5] and the fact $\psi(x) = \psi_0(x) + O(\log x)$. For $q>1$, we deduce that $\psi(x,\chi_0) = \psi(x) + O(\log x \log q)$. We note that $L(s,\chi_0)$ and $\zeta(s)$ have same zeros except those coming from the product $\prod_{p|q}(1-p^{-s})$. These are of the form $\rho = 2\pi i k/\log p$, $k \in \mathbb{Z}$. Now,
$$\sum_{p|q} \sum_{\frac{2\pi|k|}{\log p}\leq T} \left|\frac{x^\rho}{\rho}\right| \ll \sum_{p|q} \log p \log\left(\frac{T\log p}{2\pi}\right)$$
$$\ll \log q \log\left(\frac{T\log q}{2\pi}\right)$$
and the result follows immediately. □

For non-principal characters, we obtain a similar formula using the following result which is an easy consequence of [Theorem 12.10][MV06]

Lemma 2.2. *Let $T \geq 2$ and let χ be a primitive Dirichlet character modulo $q>1$. Then we have the following,*
$$\psi_0(x,\chi) = -\sum_{\substack{L(\rho,\chi)=0\\ |\mathrm{Im}(\rho)|\leq T}} \frac{x^\rho}{\rho} + C(\chi) + O\left(\log x + \frac{x}{T}(\log xqT)^2\right),$$
where $C(\chi) = \frac{L'}{L}(1,\overline{\chi}) + \log(q/2\pi) - \gamma$.

Proof. Follows immediately from [MV06, Theorem 12.10] □

We deduce the following result from the above lemma.

Lemma 2.3. *Let $T \geq 2$ and let χ be a non-principal character modulo $q>1$ induced by a primitive character χ^* modulo $q^*>1$ then,*
$$\psi(x,\chi) = -\sum_{\substack{L(\rho,\chi)=0\\ |\mathrm{Im}(\rho)|\leq T}} \frac{x^\rho}{\rho} + C(\chi^*) + O\left(\frac{x}{T}\log^2(xqT) + \log 2q \log x + \log q \log\left(\frac{T\log q}{2\pi}\right)\right)$$

Proof. We note that $\psi(x,\chi) = \psi_0(x,\chi^*) + O(\log 2q \log x)$, further we also note that that $L(s,\chi)$ and $L(s,\chi^*)$ have same zeros except for the zeros which come from the product

$\prod_{p|q, p\nmid q^*}(1-p^{-s})$. These are of the form $\rho = 2\pi i k/\log p$, $k \in \mathbb{Z}$. Now, from proof of the Lemma 2.1 we have that

$$\sum_{p|q} \sum_{\frac{2\pi|k|}{\log p} \le T} \left| \frac{x^\rho}{\rho} \right| \ll \log q \log \left(\frac{T \log q}{2\pi} \right)$$

therefore, we have proven the result. □

Following the discussion in [Dra17, Section 6] we recall the same setup to prove the preliminary results. We will use these to obtain the main results of this paper. Using Dirichlet hyperbola method [FT85, p. 45] we deduce that,

$$T(x) = 2 \sum_{q \le \sqrt{x}} (\psi(x;q,1) - \psi(q^2;q,1)) + O(\sqrt{x}\log x).$$

Therefore, we consider the sum above to find the estimates for $T(x)$. From the proof of [Dra17, Proposition 6.3] we deduce that

$$\sum_{q \le \sqrt{x}} (\psi(x;q,1) - \psi(q^2;q,1)) = S_1^+ + S_1^-,$$

where $S_1^+ = O(x(\log x)^A R^{-1/9})$ for some $A > 0$ and

$$S_1^- = \sum_{q \le \sqrt{x}} \frac{1}{\phi(q)} \sum_{\substack{\chi \bmod q \\ cond(\chi) \le R}} \sum_{q^2 < n \le x} \Lambda(n)\chi(n), \tag{2.2}$$

and $1 \le R \le x^{1/10}$ is a parameter. In his proof, Drappeau takes R to be a small power of x and proves that the contribution of non-principal characters in the sum is at most $O(x^{1-\delta})$ for some $\delta > 0$, proving the power saving error term. We consider the sum S_1^- and obtain an approximate formula for S_1^- in Section 3. We state some results which are easy consequences of Lemmas 2.1 and 2.3. In more detail, we use lemma 2.1 to deduce that for any $q \le \sqrt{x}$ and for any principal character χ_0 modulo $q \ge 1$ following holds,

$$\begin{aligned} \psi(x,\chi_0) - \psi(q^2,\chi_0) &= \sum_{q^2 < n \le x} \Lambda(n)\chi_0(n) \\ &= (x - q^2) - \sum_{\substack{L(\rho,\chi_0)=0 \\ |\mathrm{Im}(\rho)| \le T}} \frac{x^\rho - q^{2\rho}}{\rho} + O\left(\frac{x}{T} \log^2(xT) + \log^2 x \right). \end{aligned} \tag{2.3}$$

Similarly for non-principal characters, we use lemma 2.3 to conclude that for $q \le \sqrt{x}$ and any non-principal character χ modulo $q > 1$ induced by a primitive character χ^* modulo $q^* > 1$,

$$\begin{aligned} \psi(x,\chi) - \psi(q^2,\chi) &= \sum_{q^2 < n \le x} \Lambda(n)\chi(n) \\ &= - \sum_{\substack{L(\rho,\chi)=0 \\ |\mathrm{Im}(\rho)| \le T}} \frac{x^\rho - q^{2\rho}}{\rho} + O\left(\frac{x}{T} \log^2(xqT) + \log^2 x \right). \end{aligned} \tag{2.4}$$

2.1. **Results on zeros of Dirichlet L-functions.** In this subsection, we state the information we need about the zeros of Dirichlet L-functions required to prove Theorem 1.2. We need the following result.

Theorem 2.4 (Zeros results)**.** *There is an absolute and effective constant $\kappa > 0$ such that, for any $q, Q \geq 1$ the functions $F_q(s) = \prod_{\chi \bmod q} L(s,\chi)$ $G_Q(s) = \prod_{q \leq Q} \prod^*_{\chi \bmod q} L(s,\chi)$ has following properties:*

(1) (Siegel′s Theorem) *for any $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ (non-effective) such that $F_q(\sigma)$ has no real zeros $\geq 1 - C_\epsilon / q^\epsilon$.*
(2) (Page′s Theorem) *$G_Q(s)$ has at most one zero in the region $\mathrm{Re}(s) \geq 1 - \frac{\kappa}{\log(Q(2+|t|))}$. If such a zero exists it is real, simple, and arises from a real character.*
(3) (log-free zero-density estimate) *for any $\varepsilon > 0$ and any $\sigma \geq 1/2, T \geq 1$, $G_Q(s)$ has $\ll_\varepsilon (Q^2T)^{(12/5+\varepsilon)(1-\sigma)}$ zeros in the region $\mathrm{Re}(s) \geq \sigma$ and $|\mathrm{Im}(s)| \leq T$.*
(4) $(\mathrm{Vinogradov} - \mathrm{Kurobov\ zero} - \mathrm{free\ region})$ *Let $\chi \bmod q$ be a character then $L(s,\chi)$ has at most one zero in the region $\mathrm{Re}(s) \geq 1 - \frac{\kappa}{\log q + (\log|t|)^{2/3}(\log\log|t|)^{1/3}}$ and $|t| \geq 10$. If such a zero exists it is real, and arises from a real character.*

For a proof of Siegel's and Page's theorem, we refer to [MV06, Chapter 11] and the log-free zero-density estimate is proved in [Hux75] for values of σ bounded away from 1, and in [Jut77] for values of σ close to 1. For Vinogradov-Kurobov zero-free region, we refer to [Mon94, p. 176].

3. Proof of Theorem 1.1

Combining results from the previous section and results from [Dra17, Section 6] we prove the formula for $T(x)$. From equation 2.2 we know that

$$S_1^- = \sum_{q \leq \sqrt{x}} \frac{1}{\phi(q)} \sum_{\substack{\chi \bmod q \\ cond(\chi) \leq R}} \sum_{q^2 < n \leq x} \Lambda(n)\chi(n),$$

where $1 \leq R \leq x^{1/10}$ is a parameter. Using equations 2.3, 2.4 we obtain that for $T \geq 2$,

$$S_1^- = \sum_{q \leq \sqrt{x}} \frac{x - q^2}{\phi(q)} - \sum_{q \leq \sqrt{x},} \sum_{\substack{L_{R,q}(\rho)=0 \\ |\mathrm{Im}(\rho)| \leq T}} \frac{x^\rho - q^{2\rho}}{\phi(q)\rho} + O\left(R\frac{x}{T}\log^3(xT) + R\log^3 x\right).$$

We used $L_{R,q}(s)$ to denote the following product of Dirichlet L-functions $\prod_{\substack{\chi \bmod q \\ cond(\chi) \leq R}} L(s,\chi)$ in the above formula (as in the Introduction). Noting that

$$T(x) = 2(S_1^- + S_1^+) + O(\sqrt{x}\log x),$$

and using the estimate for S_1^+ from [Dra17, p. 721] and the main term from [Dra17, Theorem 1.1] we conclude that there exist $A > 0$ and $\delta \in (0,1)$ such that for $T \geq 2$ and all $x \geq 2$ we have that

$$\begin{aligned} T(x) = C_1 x\{\log x + 2\gamma - 1 - 2C_2\} - \sum_{q \leq \sqrt{x},} \sum_{\substack{L_{R,q}(\rho)=0 \\ |\mathrm{Im}(\rho)| \leq T}} \frac{2(x^\rho - q^{2\rho})}{\phi(q)\rho} \\ + O\left(R\frac{x}{T}\log^3(xT) + R\log^3 x + x(\log x)^A R^{-1/9} + x^{1-\delta}\right), \end{aligned} \tag{3.1}$$

as earlier $1 \leq R \leq x^{1/10}$ is a parameter, $C_1 = \prod_p \left(1 + \frac{1}{p(p-1)}\right)$ and $C_2 = \sum_p \left(\frac{\log p}{1+p(p-1)}\right)$. This concludes the proof of the result.

As a simple consequence of this formula, taking $R = x^{1/100}$ and $T = x^{1/10}$ we see the following result,

Corollary. *For some $\delta_1 > 0$ and all $x \geq 2$ we have that,*

$$T(x) = C_1 x\{\log x + 2\gamma - 1 - 2C_2\} - \sum_{q \leq \sqrt{x},} \sum_{\substack{L_{x^{1/100},q}(\rho)=0 \\ |\mathrm{Im}(\rho)| \leq x^{1/10}}} \frac{2(x^\rho - q^{2\rho})}{\phi(q)\rho} + O(x^{1-\delta_1}).$$

4. Proof of Theorem 1.2

We prove Theorem 1.2 using Theorem 1.1 in combination with the results about the zeros of Dirichlet L-functions stated in Theorem 2.4. Our approach will be similar to the methods used by Harper in [Har12, Section 3] to deal with sums involving characters with small conductors. We begin by proving the following lemma using zero results in Section 2. For a complex number ρ we denote by β the real part of ρ.

Lemma 4.1. *Let $x \geq 2$, suppose $0 < \eta < 1/10$ and $0 < \mu < 1$ are such that $2\eta + \mu < 1/3$ then for any primitive Dirichlet character χ mod r with $r \leq x^\eta$ we have,*

$$\sum_{\substack{L(\rho,\chi)=0, \\ |\mathrm{Im}(\rho)| \leq T}} x^{\beta - 1} = O(1),$$

where $T = x^\mu$ and the constant is absolute.

Proof. For $\alpha \geq 1/2$ and $t \geq 1$ let $N(\alpha, t)$ denote the number of zeros of $L(s, \chi)$ in the region $R(\alpha, t) := \{z \in \mathbb{C} : \mathrm{Re}(z) \geq \alpha, |\mathrm{Im}(s)| \leq t\}$. It follows from the routine calculation [IK04, Chapter 10] that

$$\sum_{\substack{L(\rho,\chi)=0, \\ |\mathrm{Im}(\rho)| \leq T}} x^{\beta-1} - 2 \leq 2 \int_{1/2}^{1} x^{\alpha-1} dN(\alpha, T)$$

$$\ll x^{-1/2} N(1/2, T) + \log x \int_{1/2}^{1} x^{\alpha-1} N(\alpha, T) d\alpha.$$

Using log-free zero-density estimates from Theorem 2.4 we conclude that

$$N(\alpha, T) \ll (x^{2\eta} T)^{3(1-\alpha)}.$$

If $2\eta + \mu < 1/3$ then we note that $x^{-1/2} N(1/2, T) \ll_B \frac{1}{\log^B x}$ for any $B > 0$. Now, using Page's theorem from Section 2 we deduce that $\int_{1/2}^{1} x^{\alpha-1} N(\alpha, T) d\alpha = \int_{\kappa'/\log x}^{1/2} \left(\frac{(x^{2\eta}T)^3}{x}\right)^\alpha d\alpha$ for some constant $\kappa' > 0$. Integrating we get,

$$\int_{\kappa'/\log x}^{1/2} \left(\frac{(x^{2\eta}T)^3}{x}\right)^\alpha d\alpha \ll \frac{1}{\log\left(\frac{(x^{2\eta}T)^3}{x}\right)} \left(\left(\frac{(x^{2\eta}T)^3}{x}\right)^{1/2} + \left(\frac{(x^{2\eta}T)^3}{x}\right)^{\kappa'/\log x}\right)$$

Again using the fact that $2\eta + \mu < 1/3$ we immediately conclude that

$$\log x \int_{1/2}^{1} x^{\alpha-1} N(\alpha, T) d\alpha = O(1).$$

The result is proven. □

We start working towards the asymptotics for $T(x+y)-T(x)$ with y in the desirable range. Considering the formula in Theorem 1.1 for the sum $T(x)$ we deduce that for $x \gg 2$, and $y \gg x^{1-\delta}$,

$$\begin{aligned}\frac{T(x+y)-T(x)}{y} =&C_1 \log x + C_1(x/y+1)\log\left(1+\frac{y}{x}\right) + \{2\gamma - 1 - 2C_2\} \\ &- \sum_{q\le\sqrt{x+y},} \sum_{\substack{L_{R,q}(\rho)=0 \\ |\mathrm{Im}(\rho)|\le T}} \frac{2((x+y)^\rho - q^{2\rho})}{\phi(q)\rho y} + \sum_{q\le\sqrt{x},} \sum_{\substack{L_{R,q}(\rho)=0 \\ |\mathrm{Im}(\rho)|\le T}} \frac{2(x^\rho - q^{2\rho})}{\phi(q)\rho y} \\ &+ O\left(R\frac{x}{Ty}\log^3(xT) + \frac{R\log^3 x}{y} + \frac{x(\log x)^A}{yR^{1/9}} + \frac{1}{\log x}\right),\end{aligned} \tag{4.1}$$

We let $R = x^\eta$ and $T = x^\mu$ such that $0 < \eta < 1/10$, $0 < \mu < 1$ and $2\eta + \mu < 1/3$. We will specify these later. Next, we consider the difference between the sums involving zeros of the Dirichlet L-functions in the above formula for $T(x+y) - T(x)$ divided by y,

$$\sum_{q\le\sqrt{x+y},} \sum_{\substack{L_{R,q}(\rho)=0 \\ |\mathrm{Im}(\rho)|\le T}} \frac{2((x+y)^\rho - q^{2\rho})}{\phi(q)\rho y} - \sum_{q\le\sqrt{x},} \sum_{\substack{L_{R,q}(\rho)=0 \\ |\mathrm{Im}(\rho)|\le T}} \frac{2(x^\rho - q^{2\rho})}{\phi(q)\rho y}.$$

For the moment we ignore the contribution coming from $\sqrt{x} < q \le \sqrt{x+y}$ in the first sum above, therefore we estimate the sum,

$$\sum_{q\le\sqrt{x},} \frac{1}{\phi(q)} \sum_{\substack{L_{R,q}(\rho)=0 \\ |\mathrm{Im}(\rho)|\le T}} \frac{2((x+y)^\rho - x^\rho)}{\rho y}.$$

Re-writing this sum over characters induced by primitive characters with small moduli we obtain that the above is the same as the sum,

$$\sum_{r\le R} \sum_{\substack{\chi^* \bmod r, \\ \chi^* \text{ primitive}}} \sum_{q\le\sqrt{x}} \frac{1}{\phi(q)} \sum_{\substack{\chi \bmod q, \\ \chi \text{ induced by } \chi^*,}} \sum_{\substack{L(\rho,\chi)=0, \\ |\mathrm{Im}(\rho)|\le T}} \frac{2((x+y)^\rho - x^\rho)}{\rho y}. \tag{4.2}$$

Let us denote by β the real part of ρ then the above summation is,

$$\ll \sum_{r\le R} \sum_{\substack{\chi^* \bmod r, \\ \chi^* \text{ primitive}}} \sum_{q\le\sqrt{x}} \frac{1}{\phi(q)} \sum_{\substack{\chi \bmod q, \\ \chi \text{ induced by } \chi^*,}} \sum_{\substack{L(\rho,\chi)=0, \\ |\mathrm{Im}(\rho)|\le T}} x^{\beta-1}.$$

If χ mod q is induced by a primitive character χ^* mod r then $L(s,\chi)$ and $L(s,\chi^*)$ have same zeros except the zeros of the product $\prod_{p|q, p\nmid r}(1-p^{-s})$, but R is a small power of x, therefore for any $B > 0$ the expression we are considering above is,

$$\sum_{r\le R} \sum_{\substack{\chi^* \bmod r, \\ \chi^* \text{ primitive}}} \sum_{\substack{q\le\sqrt{x}, \\ r|q}} \frac{1}{\phi(q)} \sum_{\substack{L(\rho,\chi^*)=0, \\ |\mathrm{Im}(\rho)|\le T}} x^{\beta-1} + O_B\left(\frac{1}{\log^B x}\right).$$

Now, we focus on the simpler sum,

$$\sum_{r\le R}\sum_{\substack{\chi^* \bmod r,\\ \chi^* \text{ primitive}}}\sum_{\substack{q\le\sqrt{x},\\ r|q}}\frac{1}{\phi(q)}\sum_{\substack{L(\rho,\chi^*)=0,\\ |\mathrm{Im}(\rho)|\le T}} x^{\beta-1}.$$

We will deal with this sum by dividing it into multiple pieces. We define the following "good" set of characters, making this sum easier to handle. Let $c_1, c_2 > 0$ we will specify them later to obtain the desired asymptotics,

$$\mathcal{S} := \bigcup_{1<r\le e^{\eta\sqrt{\log x}}}\{\chi^* \bmod r, \text{ primitive} : L(s,\chi^*)\neq 0 \text{ for any } \mathrm{Re}(s) > 1-\frac{c_1}{(\log x)^{3/4}},$$
$$|\mathrm{Im}(s)|\le T\},$$

and

$$\mathcal{S}_2 := \bigcup_{e^{\eta\sqrt{\log x}}<r\le R}\{\chi^* \bmod r,\ \text{primitive} : L(s,\chi^*)\neq 0 \text{ for any } \mathrm{Re}(s) > 1-\frac{c_2}{\sqrt{\log x}},$$
$$|\mathrm{Im}(s)|\le T\}.$$

Suppose $\chi^* \in \mathcal{S}_1$ and let $N(\alpha, t)$ be defined as in the proof of Lemma 4.1. Then as in the proof of the Lemma 4.1 using log-free zero-density estimate and the definition of $\mathcal{S}_1$ we obtain the following after computation

$$\begin{aligned}\sum_{\substack{L(\rho,\chi^*)=0,\\ |\mathrm{Im}(\rho)|\le T}} x^{\beta-1} &\le 2\int_{1/2}^{1} x^{\alpha-1}dN(\alpha,T)\\ &\ll x^{-1/2}N(1/2,T)+\log x\int_{1/2}^{1}x^{\alpha-1}N(\alpha,T)d\alpha\\ &\ll x^{-1/2}N(1/2,T)+\log x\int_{1-c_1/(\log x)^{3/4}}^{1}x^{\alpha-1}N(\alpha,T)d\alpha\\ &\ll_B \frac{1}{\log^B x},\end{aligned}$$

for any $B > 0$. For $\chi^* \in \mathcal{S}_2$, we proceed by splitting the sum into dyadic intervals. For any $3 \le M \le x^{\eta}$, let $N_M(\alpha, t)$ denote the number of zeros of $G_{2M}(s)$ (see Theorem 2.4) in the region $R(\alpha, t) := \{z \in \mathbb{C} : \mathrm{Re}(z) \ge \alpha, |\mathrm{Im}(s)| \le t\}$. From log-free zero-density estimate we get the bound $N_M(\alpha, T) \ll (M^2T)^{3(1-\alpha)}$. Using this bound we conclude that,

$$\begin{aligned}\sum_{M<r\le 2M}\sum_{\substack{\chi^* \bmod r,\\ \chi*\in\mathcal{S}_2}}\frac{1}{\phi(r)}\sum_{\substack{L(\rho,\chi^*)=0,\\ |\mathrm{Im}(\rho)|\le T}} x^{\beta-1} &\ll \frac{\log\log M}{M}\sum_{M<r\le 2M}\sum_{\substack{\chi^* \bmod r,\\ \chi*\in\mathcal{S}_2}}\sum_{\substack{L(\rho,\chi^*)=0,\\ |\mathrm{Im}(\rho)|\le T}} x^{\beta-1}\\ &\ll \frac{\log\log M}{M}\int_{1/2}^{1}x^{\alpha-1}dN_M(\alpha,T)\\ &\ll \frac{\log\log M}{M},\end{aligned}$$

as the bound on $N_M(\alpha, t)$ gives us the following estimate,

$$\int_{1/2}^{1} x^{\alpha-1} dN_M(\alpha, T) \ll x^{-1/2} N_M(1/2, T) + \log x \int_{c_2/\sqrt{\log x}}^{1/2} \left(\frac{x^{3(2\eta+\mu)}}{x} \right)^{\alpha} d\alpha \\ \ll 1.$$

Summing by splitting into dyadic intervals, we note that we need to divide the interval $(e^{\eta\sqrt{\log x}}, x^{\eta}]$ into at most $\ll \log x$ dyadic intervals, and using the above estimate we obtain,

$$\sum_{r \le R,} \sum_{\substack{\chi^* \bmod r, \\ \chi^* \in \mathcal{S}_2}} \sum_{\substack{q \le \sqrt{x}, \\ r|q}} \frac{1}{\phi(q)} \sum_{\substack{L(\rho, \chi^*)=0, \\ |\mathrm{Im}(\rho)| \le T}} x^{\beta-1} \ll_B \frac{1}{\log^B x},$$

for any $B > 0$. Next, we work with primitive characters with moduli r in the range $e^{\eta\sqrt{\log x}} x < r \le R$ but which are not in $\mathcal{S}_2$. We will follow the same strategy of splitting into dyadic intervals. Using log-free zero-density estimate from Section 2 we obtain that that for any $M \ge 3$,

$$\sum_{M < r \le 2M} \sum_{\substack{\chi^* \bmod r, \\ \chi * \notin \mathcal{S}_2}} \frac{1}{\phi(r)} \ll \frac{\log\log M}{M} \sum_{M < r \le 2M} \sum_{\substack{\chi^* \bmod r, \\ L(s,\chi^*)=0 \text{ for some } s \\ \mathrm{Re}(s) > 1 - c_2/\sqrt{\log x}, |\mathrm{Im}(s)| \le (2M)^{\mu\sqrt{\log x}/\eta}}} 1 \\ \ll \frac{\log\log M}{M} (M^{2/\sqrt{\log x}} (2M)^{\mu/\eta})^{3c_2}.$$

Assuming $\mu/\eta < 1000$ and choosing c_2 in a suitable manner we can assume that the above is $\ll M^{-1/17}$. Similar to the sum for characters in $\mathcal{S}_2$ we again sum by splitting into dyadic intervals, and using Lemma 4.1 we obtain,

$$\sum_{r \le R,} \sum_{\substack{\chi^* \bmod r, \\ \chi^* \notin \mathcal{S}_2}} \sum_{\substack{q \le \sqrt{x}, \\ r|q}} \frac{1}{\phi(q)} \sum_{\substack{L(\rho, \chi^*)=0, \\ |\mathrm{Im}(\rho)| \le T}} x^{\beta-1} \ll_B \frac{1}{\log^B x},$$

for any $B > 0$. Now, we only have to deal with contributions from characters with conductors less than or equal to $e^{\eta\sqrt{\log x}}$ and not in $\mathcal{S}_1$. Using Vinogradov-Korobov zero free region given in Theorem 2.4 we can choose c_1 such that there can be at most one such bad character with only one real zero. Let us call this character χ^*_{bad} with conductor r_{bad}, its contribution to sum of our interest is,

$$\sum_{\substack{q \le \sqrt{x}, \\ r_{bad}|q}} \frac{1}{\phi(q)} \sum_{\substack{L(\rho, \chi^*_{bad})=0, \\ |\mathrm{Im}(\rho)| \le T}} x^{\beta-1} \ll \frac{1}{\phi(r_{bad})} \sum_{q \le x/r_{bad}} \frac{1}{\phi(q)} \\ \ll \frac{\log x}{\phi(r_{bad})}.$$

As $L(s, \chi^*_{bad})$ has a real zero greater than $1 - c_1/\sqrt{\log x}$, Siegel's theorem from Section 2 implies that $r_{bad} \gg_B \log^B x$ for any $B > 0$. As a consequence we have,

$$\sum_{\substack{q \le \sqrt{x}, \\ r_{bad}|q}} \frac{1}{\phi(q)} \sum_{\substack{L(\rho, \chi^*_{bad})=0, \\ |\mathrm{Im}(\rho)| \le T}} x^{\beta-1} \ll_B \frac{1}{\log^B x}$$

for all $B > 0$. Combining all the previous results for characters in different sets we considered, we conclude that,

$$\sum_{r\leq R}\sum_{\substack{\chi^* \bmod r,\\ \chi^* \text{ primitive}}}\sum_{\substack{q\leq\sqrt{x},\\ r|q}}\frac{1}{\phi(q)}\sum_{\substack{L(\rho,\chi^*)=0,\\ |\mathrm{Im}(\rho)|\leq T}} x^{\beta-1} = O_B\left(\frac{1}{\log^B x}\right).$$

from an earlier discussion, we see that the same estimate is true for the expression 4.2. Using the mean value theorem and along with the discussion in this section we see that summands with $\sqrt{x} < q \leq \sqrt{x+y}$ contribute at most $O_B(1/\log^B x)$ for all $B > 0$. From the error term in the equation 4.1 and proof of Hoheisel's theorem in [IK04, Chapter 10] we see that there exists a θ in $(0,1)$ such that the following holds for $x \gg 2$ and $x^\theta < y < x$,

$$T(x+y) - T(x) = C_1 y\log x + C_1(x+y)\log\left(1+\frac{y}{x}\right) + \{2\gamma - 1 - 2C_2\}y + O\left(\frac{y}{\log x}\right).$$

This concludes the proof of Theorem 1.2. Next, we present a simple corollary.

Corollary. *Let θ and y be as in the Theorem 1.2, suppose that $\lim_{x\to\infty} y/x = 0$ then we have that,*

$$\lim_{x\to\infty}\frac{T(x+y)-T(x)}{y} - C_1\log x = C_1 - 2C_2 + 2\gamma - 1.$$

Acknowledgements

I would like to express my sincere gratitude to my advisor, Prof. Sary Drappeau, for his invaluable guidance and support throughout my project at the Institut de Mathématiques de Marseille. Their expertise and encouragement were instrumental in shaping this write-up.

This work was done at the Institut de Mathématiques de Marseille. The author was supported by the Institut de Mathématiques de Marseille and ReLaX lab during his stay in Marseille.

References

[ABL21] E. Assing, V. Blomer, and J. Li. Uniform Titchmarsh divisor problems. *Advances in Mathematics*, 393:108076, 2021.

[BFI86] E. Bombieri, J. B. Friedlander, and H. Iwaniec. Primes in arithmetic progressions to large moduli. *Acta Math.*, 156:203–251, 1986.

[Dra15] Sary Drappeau. Théorèmes de type Fouvry–Iwaniec pour les entiers friables. *Compositio Mathematica*, 151(5):828–862, 2015.

[Dra17] S. Drappeau. Sums of Kloosterman sums in arithmetic progressions, and the error term in the dispersion method. *Proceedings of the London Mathematical Society*, 114(4):684–732, Feb 2017.

[DT19] S. Drappeau and B. Topacogullari. Combinatorial identities and Titchmarsh's divisor problem for multiplicative functions. *Algebra & Number Theory*, 13(10):2383–2425, Dec 2019.

[Fou85] E. Fouvry. Sur le problème des diviseurs de Titchmarsh. *J. Reine Angew. Math.*, 357:51–76, 1985.

[FT85] É. Fouvry and G. Tenenbaum. Sur la corrélation des fonctions de Piltz. *Rev. Mat. Iberoam*, 1(3):43–54, Sep 1985.

[FT90] E. Fouvry and G. Tenenbaum. Diviseurs de Titchmarsh des entiers sans grand facteur premier. In Kenji Nagasaka and Etienne Fouvry, editors, *Analytic Number Theory*, pages 86–102, Berlin, Heidelberg, 1990. Springer Berlin Heidelberg.

[Hal67] H. Halberstam. Footnote to the Titchmarsh-Linnik divisor problem. *Proc. Amer. Math. Soc.*, 18:187–188, 1967.

[Har12] Adam J. Harper. Bombieri–Vinogradov and Barban–Davenport–Halberstam type theorems for smooth numbers, 2012.

[Hux75] M. Huxley. Large values of Dirichlet polynomials, III. *Acta Arithmetica*, 26(4):435–444, 1975.

[IK04] H. Iwaniec and E. Kowalski. *Analytic Number Theory*, volume 53 of *Colloquium Publications*. American Mathematical Society, 2004.

[Jut77] M. Jutila. On Linnik's constant. *Mathematica Scandinavica*, 41(1):45–62, 1977.

[Lin63] Ju. V. Linnik. *The dispersion method in binary additive problems*. Translated by S. Schur (American Mathematical Society, Providence, RI), 1963.

[Mon94] H. L. Montgomery. *Ten Lectures on the Interface between Analytic Number Theory and Harmonic Analysis*, volume 84 of *CBMS Regional Conference Series in Mathematics*. American Mathematical Society, 1994.

[Mot80] Y. Motohashi. An asymptotic series for an additive divisor problem. *Math. Z.*, 170:43–63, 1980.

[MV06] H. L. Montgomery and R. C. Vaughan. *Multiplicative Number Theory I: Classical Theory*. Cambridge University Press, 2006.

[Rod65] G. Rodriquez. Sul problema dei divisori di Titchmarsh. *Boll. Un. Mat. Ital.*, 20(3):358–366, 1965.

[Tao16] Terence Tao. The logarithmically averaged Chowla and Elliott conjectures for two-point correlations. *Forum of Mathematics, Pi*, 4:e8, 2016.

[Tit30] E. C. Titchmarsh. A divisor problem. *Rend. Circ. Mat. Palermo.*, 54(2):414–429, 1930.

[Top16] B. Topacogullari. The Shifted Convolution of divisor functions. *The Quarterly Journal of Mathematics*, 67(2):331–363, 05 2016.

[Top17] B. Topacogullari. The Shifted Convolution of Generalized Divisor Functions. *International Mathematics Research Notices*, 2018(24):7681–7724, 06 2017.

Chennai Mathematical Institute, H1, SIPCOT IT Park, Kelambakkam, Siruseri, Tamil Nadu 603103

Email address: hrishabh@cmi.ac.in